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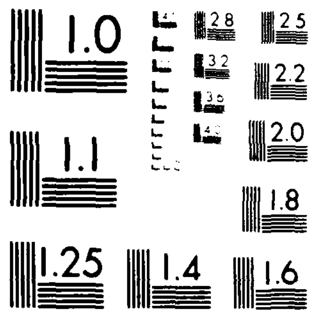

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To location L_i we are to allocate a 'generator' and n_i 'machines' for $i=1,\ldots, k$ where $n_1 \geq \cdots \geq n_k$. Although the generators and machines function independently of one another, a machine is operable only if it and the generator at its location are functioning. The problem the authors consider is that of finding the arrangement or allocation optimizing the number of operable machines. The

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ITEM #20, CONTINUED: authors show that if the objective is to maximize the expected number of operable machines at some future time, then it is best to allocate the best generator and the n_1 best machines to location L_1, the 2^{nd} best generator and the n_2 next best machines to location L_2, etc. However, this arrangement is not always stochastically optimal. For the case of 2 generators the authors give a necessary and sufficient condition that this arrangement is stochastically best, and illustrate the result with several examples.

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Optimal Arrangement of Systems

by

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Optimal Arrangement of Systems

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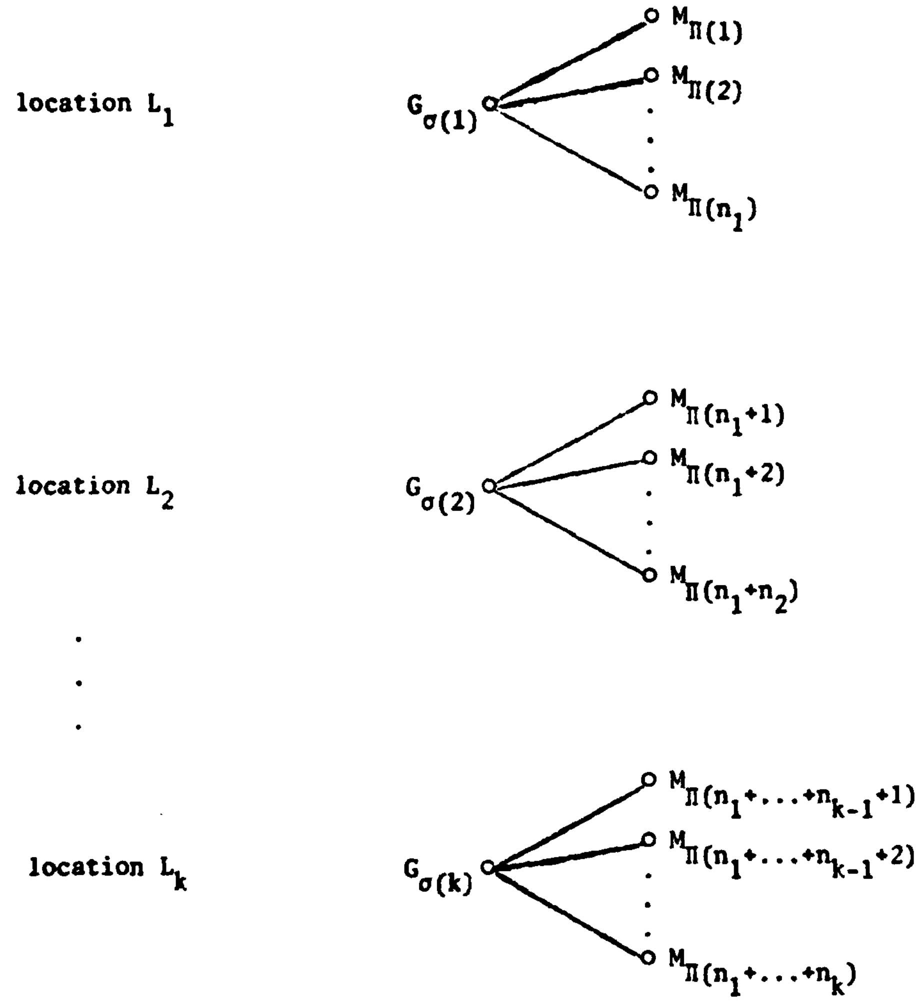
Abstract

To location L_i we are to allocate a "generator" and n_i "machines" for $i = 1, \ldots, k$ where $n_1 \geq \ldots \geq n_k$. Although the generators and machines function independently of one another, a machine is operable only if it and the generator at its location are functioning. The problem we consider is that of finding the arrangement or allocation optimizing the number of operable machines. We show that if the objective is to maximize the expected number of operable machines at some future time, then it is best to allocate the best generator and the n_1 best machines to location L_1, the 2^{nd} best generator and the n_2 next best machines to location L_2, etc. However this arrangement is not always stochastically optimal. For the case of 2 generators we give a necessary and sufficient condition that this arrangement is stochastically best, and illustrate the result with several examples.

Introduction.

Machines M_1, M_2, ..., $M_{n_1+\cdots+n_k}$ of a similar type are to be connected to k generators G_1, ..., G_k. We assume that $n_1 \geq n_2 \geq \ldots \geq n_k$ and that n_i machines and a generator are to be allocated to location L_i for i = 1, ..., k. All of the machines at a particular location are connected to the generator there, and although all generators and machines function independently, a machine will be termed operable only if both it and the generator to which it is connected are functioning. We let $p_i(p_{2j})$ be the probability that machine i (generator j) is functioning at some specified time t_0 in the future. Let $X_i(X_{2j})$ be the indicator random variable which is 1 if machine i (generator j) is functioning at time t_0 and 0 otherwise. For any permutations σ of $\{1, 2, \ldots, k\}$ and Π of $\{1, \ldots, n_1 + n_2 + \ldots + n_k\}$ we let A_σ^Π represent the allocation or arrangement whereby machines $M_{\Pi(n_1+\ldots n_{i-1}+1)}, \ldots, M_{\Pi(n_1+\ldots+n_i)}$ and generator $G_{\sigma(i)}$ are allocated to location L_i for i = 1, ..., k.

<u>Arrangement</u> A_σ^Π



N_σ^Π will be the random variable indicating the number of operable machines at time t_0 when using arrangement A_σ^Π. Hence

$$N_\sigma^\Pi = X_{2\sigma(1)}(X_{\Pi(1)}+\ldots+X_{\Pi(n_1)}) +$$

$$\ldots+ X_{2\sigma(k)}(X_{\Pi(n_1+\ldots+n_{k-1}+1)}+\ldots+ X_{\Pi(n_1+\ldots+n_k)}).$$

When Π (respectively σ) is the identity permutation we drop the symbol $\Pi(\sigma)$ in the notation N_σ^Π. For example

$$N = X_{21}(X_1+\ldots+X_{n_1})+\ldots+X_{2k}(X_{n_1+\ldots+n_{k-1}+1}+\ldots+X_{n_1+\ldots+n_k}).$$

Without loss of generality we assume that the generators and machines have been labelled so that $p_{21} \geq p_{22} \geq \cdots \geq p_{2k}$ and $p_1 \geq p_2 \geq \cdots \geq p_{n_1+\ldots+n_k}$.

The problem we consider is that of determining the arrangement A_σ^Π which in some sense "optimizes" the number N_σ^Π of operable machines at time t_0. We show in Section 1 that N is always optimal in the sense of maximizing the expected number of operable machines at time t_0. That is, the optimal arrangement is to allocate the best generator and the n_1 best machines to location L_1, the 2nd best generator and the next n_2 best machines to location L_2, etc. Although $E(N) \geq E(N_\sigma^\Pi)$ for all Π and σ, it is however not true that in general $N \overset{st}{\geq} N_\sigma^\Pi$ (N is stochastically larger than N_σ^Π) for all Π and σ. In Section 2 we investigate the situation of 2 generators ($k = 2$), and we show for example that when $p_1 \geq \cdots \geq p_{n_1} > p_{n_1+1} \geq \cdots \geq p_{n_1+n_2} \geq \frac{1}{2}$, a necessary and sufficient condition for $N \overset{st}{\geq} N_\sigma^\Pi$ for all Π and σ is that

$$\left(\frac{p_{21}}{1-p_{21}}\right)\Big/\left(\frac{p_{22}}{1-p_{22}}\right) \geq \frac{q_{n_1+2}\cdots q_{n_1+n_2}}{q_1\cdots q_{n_1-1}} \qquad \text{where } q = 1 - p.$$

Such a characterization is of considerable interest, for when $N \overset{st}{\geq} N_\sigma^\Pi$ for all Π and σ, N clearly represents the optimal arrangement in every sense of the word.

There are of course many variations of this problem. Instead of the terminology "machines" and "generators" we may consider for example telephones and switchboards, or computer terminals and computers, or speakers and amplifiers. Although our "machines" or "generators" are usually of the same type — that is to say they have a similar life distribution — they might be of different ages which would enable us to rank them according to the probability of their functioning at some specific time in the future. Also more generally we could consider problems with more than two "stages" (for example a three "stage" problem involving "generators", "power relay mechanisms", and "machines").

For results of a related nature, see Derman, Lieberman, and Ross [1972] and [1974].

1. Optimizing the Expected Number of Operable Machines.

We begin by proving some elementary inequalities.

Lemma 1.1. Let $p_{21} \geq p_{22} \geq 0$ and $p_1 \geq p_2 \geq \cdots \geq p_{n_1+n_2} \geq 0$ where $n_1 \geq n_2$. If σ and Π are arbitrary permutations on $\{1, 2\}$ and $\{1, 2, \ldots, n_1 + n_2\}$ respectively, then

$$p_{21}(p_1+\cdots+p_{n_1}) + p_{22}(p_{n_1+1}+\cdots+p_{n_1+n_2})$$

$$\geq p_{2\sigma(1)}(p_{\Pi(1)}+\cdots+p_{\Pi(n_1)})+p_{2\sigma(2)}(p_{\Pi(n_1+1)}+\cdots+p_{\Pi(n_1+n_2)}). \quad (1)$$

Proof. a) We consider the case where $\sigma(1) = 1$. Defining $U = \{1, \ldots, n_1\}/\{\Pi(1), \ldots, \Pi(n_1)\}$ and $V = \{\Pi(1),\ldots,\Pi(n_1)\}/\{1,\ldots,n_1\}$ we see that $|U| = |V|$ and moreover that $p_i \geq p_j$ whenever $i \in U$ and $j \in V$. Therefore

$$p_{21}\Big(\sum_{i\in U} p_i - \sum_{j\in V} p_j\Big) \geq p_{22}\Big(\sum_{i\in U} p_i - \sum_{j\in V} p_j\Big)$$

from which (1) follows.

b) Suppose now that $\sigma(1) = 2$. Now

$$p_1+\ldots+p_{n_1}-(p_{\Pi(n_1+1)}+\ldots+p_{\Pi(n_1+n_2)}) =$$

$$p_{\Pi(1)}+\ldots+p_{\Pi(n_1)}-(p_{n_1+1}+\ldots+p_{n_1+n_2})$$

and each of these two (equal) expressions are ≥ 0 since $n_1 \geq n_2$ and the p_i's are nonincreasing. Multiplying on the left by p_{21} and on the right by p_{22} $(\leq p_{21})$ and transforming we obtain (1). ||

Using Lemma 1.1, we may prove the following extension.

Lemma 1.2. Let $p_{21} \geq \cdots \geq p_{2k} \geq 0$ and $p_1 \geq \ldots \geq p_{n_1+\ldots+n_k} \geq 0$ where $n_1 \geq n_2 \geq \ldots \geq n_k$. If σ and Π are arbitrary permutations on $\{1, \ldots, k\}$ and $\{1, \ldots, n_1+\ldots+n_k\}$ respectively, then

$$\sum_{i=1}^{k} p_{2i}\left(\sum_{j=n_1+\ldots+n_{i-1}+1}^{n_1+\ldots+n_i}\right) \geq \sum_{i=1}^{k} p_{2\sigma(i)}\left(\sum_{j=n_1+\ldots+n_{i-1}+1}^{n_1+\ldots+n_i} p_{\Pi(j)}\right) .$$

Theorem 1.3. $E(N) \geq E(N_\sigma^\Pi)$ for all permutations σ and Π of $\{1, \ldots, k\}$ and $\{1, \ldots, n_1+\ldots+n_k\}$ respectively.

Proof. We are assuming that generators and machines function independently of one another and hence $E(X_{2j}X_i) = p_{2j}p_i$ for any j and i. Therefore given σ and Π,

$$E(N_\sigma^\Pi) = E\left(\sum_{i=1}^{k} X_{2\sigma(i)}\left[\sum_{j=n_1+\dots+n_{i-1}+1}^{n_1+\dots+n_i} X_{\Pi(j)}\right]\right)$$

$$= \sum_{i=1}^{k} p_{2\sigma(i)} \left(\sum_{j=n_1+\dots+n_{i-1}+1}^{n_1+\dots+n_i} p_{\Pi(j)}\right) ,$$

and hence the theorem follows from Lemma 1.2. ||

Application 1.4. Theorem 1.3 implies that if our criterion is to maximize the expected number of operable machines at some time t_0 in the future, then the optimal policy is: Determine which location needs the most (n_1) machines, and then allocate the best generator and n_1 best machines to that location. Next find the location needing the next largest number (n_2) of machines. Allocate to this location the 2nd best generator and the next n_2 best machines. Continue in this fashion.

Remark 1.5. It should be clear that generalizations of Theorem 1.3 can be made to problems with more than two "stages", although we do not give details here.

2. Stochastic Optimization of N.

We assume in this section unless otherwise stated that we are dealing with 2(k = 2) generators, and for ease of notation write $n = n_1$ and $m = n_2$ $(n \geq m)$. Initially we confine ourselves to arrangements of the form A^{Π}, that is where the best generator is allocated to the location L_1 needing the most machines (n).

Given a specific permutation Π of $\{1, \ldots, n, \ldots, n + m\}$ we can without loss of generality assume that $\Pi(1) < \ldots < \Pi(n)$ and $\Pi(n+1) < \ldots < \Pi(n+m)$. If $\Pi(n+1) < \Pi(n)$ (otherwise Π = identity), we define Π' by $\Pi'(i) = \Pi(i)$ for $i \notin \{n, n+1\}$, $\Pi'(n) = \Pi(n+1)$, and $\Pi'(n+1) = \Pi(n)$, We now investigate conditions under which $N^{\Pi'}$ is stochastically superior to N^{Π} (i.e., $N^{\Pi'} \overset{st}{\geq} N^{\Pi}$).

If E is an event in a probability space, we use the notation Probability (E) = P(E) = [E].

<u>Lemma 2.1</u>. Let $p_{21} \geq p_{22} \geq 0$. For $1 \leq r \leq n$, $P[N^{\Pi'} \geq r] \geq P[N^{\Pi} \geq r]$ if and only if

$$\left(\frac{p_{21}}{1-p_{21}}\right) \Big/ \left(\frac{p_{22}}{1-p_{22}}\right) \geq \frac{[X_{\Pi(n+2)}+\ldots+X_{\Pi(n+m)}=r-1]}{[X_{\Pi(1)}+\ldots+X_{\Pi(n-1)}=r-1]} .$$

<u>Proof</u>. If $p_{\Pi(n)} = p_{\Pi(n+1)}$, then $P[N^{\Pi'} \geq r] = P[N^{\Pi} \geq r]$ and

$$\frac{[X_{\Pi(n+2)}+\ldots+X_{\Pi(n+m)}=r-1]}{[X_{\Pi(1)}+\ldots+X_{\Pi(n-1)}=r-1]} \leq 1,$$ and so the result is true.

Hence without loss of generality we may assume $P_{\Pi(n)} < P_{\Pi(n+1)}$.

Now

$$[N^{\Pi'} \geq r] = p_{21}p_{22}[\sum_{1}^{n+m} X_i \geq r] + p_{21}(1-p_{22})[X_{\Pi(1)}+\cdots+X_{\Pi(n-1)}+X_{\Pi(n+1)} \geq r]$$

$$+ p_{22}(1-p_{21})[X_{\Pi(n)}+X_{\Pi(n+2)}+\cdots+X_{\Pi(n+m)} \geq r]$$

$$\geq p_{21}p_{22}[\sum_{1}^{n+m} X_i \geq r] + p_{21}(1-p_{22})[X_{\Pi(1)}+\cdots+X_{\Pi(n-1)}+X_{\Pi(n)} \geq r]$$

$$+ p_{22}(1-p_{21})[X_{\Pi(n+1)} + X_{\Pi(n+2)}+\cdots+X_{\Pi(n+m)} \geq r]$$

$$= [N^{\Pi} \geq r]$$

$\iff$

$$p_{21}(1-p_{22})\{[X_{\Pi(1)}+\cdots+X_{\Pi(n-1)}+X_{\Pi(n+1)} \geq r] - [X_{\Pi(1)}+\cdots+X_{\Pi(n-1)}+X_{\Pi(n)} \geq r]\}$$

$$\geq p_{22}(1-p_{21})\{[X_{\Pi(n+1)}+\cdots+X_{\Pi(n+m)} \geq r] - [X_{\Pi(n)}+X_{\Pi(n+2)}+\cdots+X_{\Pi(n+m)} \geq r]\}$$

$\iff$ (since $p_{\Pi(n+1)} > p_{\Pi(n)}$). Thus

$$\left(\frac{p_{21}}{1-p_{21}}\right)\Big/\left(\frac{p_{22}}{1-p_{22}}\right) \geq \frac{[X_{\Pi(n+1)}+\cdots+X_{\Pi(n+m)} \geq r] - [X_{\Pi(n)}+X_{\Pi(n+2)}+\cdots+X_{\Pi(n+m)} \geq r]}{[X_{\Pi(1)}+\cdots+X_{\Pi(n-1)}+X_{\Pi(n+1)} \geq r] - [X_{\Pi(1)}+\cdots+X_{\Pi(n)} \geq r]}$$

$$= \{p_{\Pi(n+1)}[X_{\Pi(n+2)}+\cdots+X_{\Pi(n+m)} \geq r-1] + q_{\Pi(n+1)}[X_{\Pi(n+2)}+\cdots+X_{\Pi(n+m)} \geq r]$$

$$- p_{\Pi(n)}[X_{\Pi(n+2)}+\cdots+X_{\Pi(n+m)} \geq r-1] - q_{\Pi(n)}[X_{\Pi(n+2)}+\cdots+X_{\Pi(n+m)} \geq r]\}\Big/$$

$$\{p_{\Pi(n+1)}[X_{\Pi(1)}+\cdots+X_{\Pi(n-1)} \geq r-1] + q_{\Pi(n+1)}[X_{\Pi(1)}+\cdots+X_{\Pi(n-1)} \geq r] -$$

$$- p_{\Pi(n)}[X_{\Pi(1)}+\cdots+X_{\Pi(n-1)} \geq r-1] - q_{\Pi(n)}[X_{\Pi(1)}+\cdots+X_{\Pi(n-1)} \geq r]\}.$$

$$= \frac{(p_{\Pi(n+1)}-p_{\Pi(n)})\{[X_{\Pi(n+2)}+\ldots+X_{\Pi(n+m)}\geq r-1]-[X_{\Pi(n+2)}+\ldots+X_{\Pi(n+m)}\geq r]\}}{(p_{\Pi(n+1)}-p_{\Pi(n)})\{[X_{\Pi(1)}+\ldots+X_{\Pi(n-1)}\geq r-1]-[X_{\Pi(1)}+\ldots+X_{\Pi(n-1)}\geq r]\}}$$

$$= \frac{[X_{\Pi(n+2)}+\ldots+X_{\Pi(n+m)}=r-1]}{[X_{\Pi(1)}+\ldots+X_{\Pi(n-1)}=r-1]} \;. \;\|$$

Remark 2.2. Note that if $r = 0$ or $r > n$, then $P(N^{\Pi'}\geq r)=P(N^{\Pi}\geq r)$.

Lemma 2.3. For $1 \leq r \leq n$,

$$\frac{[X_{\Pi(n+2)}+\ldots+X_{\Pi(n+m)}=r-1]}{[X_{\Pi(1)}+\ldots+X_{\Pi(n-1)}=r-1]} \leq \frac{p_{\Pi(n+2)}\cdots p_{\Pi(n+r)}q_{\Pi(n+r+1)}\cdots q_{\Pi(n+m)}\binom{n-1}{r-1}}{p_{\Pi(n-r+1)}\cdots p_{\Pi(n-1)}q_{\Pi(1)}\cdots q_{\Pi(n-r)}\binom{n-1}{r-1}}$$

Proof. In what follows, ε_j will denote a binary variable taking the value 0 or 1.

$$\frac{[X_{\Pi(n+2)}+\ldots+X_{\Pi(n+m)}=r-1]}{[X_{\Pi(1)}+\ldots+X_{\Pi(n-1)}=r-1]}$$

$$= \frac{\sum_{\varepsilon_{n+2}+\ldots+\varepsilon_{n+m}=r-1} p_{\Pi(n+2)}^{\varepsilon_{n+2}}\cdots p_{\Pi(n+m)}^{\varepsilon_{n+m}} q_{\Pi(n+2)}^{1-\varepsilon_{n+2}}\cdots q_{\Pi(n+m)}^{1-\varepsilon_{n+m}}}{\sum_{\varepsilon_1+\ldots+\varepsilon_{n-1}=r-1} p_{\Pi(1)}^{\varepsilon_1}\cdots p_{\Pi(n-1)}^{\varepsilon_{n-1}} q_{\Pi(1)}^{1-\varepsilon_1}\cdots q_{\Pi(n-1)}^{1-\varepsilon_{n-1}}}$$

As the p_i's are nonincreasing in i, it follows that

$$p_{\Pi(n+2)}^{\varepsilon_{n+2}}\cdots p_{\Pi(n+m)}^{\varepsilon_{n+m}} q_{\Pi(n+2)}^{1-\varepsilon_{n+2}}\cdots q_{\Pi(n+m)}^{1-\varepsilon_{n+m}} \leq p_{\Pi(n+2)}\cdots p_{\Pi(n+r)}\, q_{\Pi(n+r+1)}\cdots q_{\Pi(n+m)}$$

and

$$p_{\Pi(1)}^{\epsilon_1} \cdots p_{\Pi(n-1)}^{\epsilon_{n-1}} q_{\Pi(1)}^{1-\epsilon_1} \cdots q_{\Pi(n-1)}^{1-\epsilon_{n-1}} \geq q_{\Pi(1)} \cdots q_{\Pi(n-r)} p_{\Pi(n-r+1)} \cdots p_{\Pi(n-1)} .$$

Hence

$$\frac{[X_{\Pi(n+2)}+\cdots+X_{\Pi(n+m)}=r-1]}{[X_{\Pi(1)}+\cdots+X_{\Pi(n-1)}=r-1]} < \frac{\binom{m-1}{r-1} p_{\Pi(n+2)}\cdots p_{\Pi(n+r)} q_{\Pi(n+r+1)}\cdots q_{\Pi(n+m)}}{\binom{n-1}{r-1} p_{\Pi(n-r+1)}\cdots p_{\Pi(n-1)} q_{\Pi(1)}\cdots q_{\Pi(n-r)}} . \quad ||$$

<u>Lemma 2.4</u>. Assume that $p_1 \geq \ldots \geq p_{n+m} \geq \frac{1}{2}$ and that $1 \leq r \leq n$. Then

$$\frac{\binom{m-1}{r-1} p_{\Pi(n+2)}\cdots p_{\Pi(n+r)} q_{\Pi(n+r+1)}\cdots q_{\Pi(n+m)}}{\binom{n-1}{r-1} p_{\Pi(n-r+1)}\cdots p_{\Pi(n-1)} q_{\Pi(1)}\cdots q_{\Pi(n-r)}}$$

$$\leq \frac{\binom{m-1}{r-1} p_{n-r+1}\cdots p_{n-1} q_{n+r+1}\cdots q_{n+m}}{\binom{n-1}{r-1} p_{n+2}\cdots p_{n+r} q_1\cdots q_{n-r}} \equiv C_r .$$

<u>Proof</u>. Since p_i is nonincreasing in i and $p_i \geq \frac{1}{2}$, $p_i q_i \leq p_{i+1} q_{i+1}$ for all $i = 1, \ldots, n+m-1$. We may therefore obtain an upper bound for

$$\frac{p_{\Pi(n+2)}\cdots p_{\Pi(n+r)} q_{\Pi(n+r+1)}\cdots q_{\Pi(n+m)}}{p_{\Pi(n-r+1)}\cdots p_{\Pi(n-1)} q_{\Pi(1)}\cdots q_{\Pi(n-r)}}$$

by arguing that we may assume that every index of q in the numerator is > every index of p in the denominator, which in turn is > every index of p in the numerator and which in turn is > every index of q in the denominator, from which the result follows. ||

Lemma 2.5. Assume $p_1 \geq \dots \geq p_{n+m} \geq \frac{1}{2}$ and that $1 \leq r \leq n$. Then

$$C_r \equiv \frac{\binom{m-1}{r-1} p_{n-r+1} \cdots p_{n-1} q_{n+r+1} \cdots q_{n+m}}{\binom{n-1}{r-1} p_{n+2} \cdots p_{n+r} q_1 \cdots q_{n-r}} \text{ is } \downarrow \text{ in } r.$$

Proof. Note that $C_r = 0$ for $r > m$ since in this case $\binom{m-1}{r-1} = 0$. It is easy to verify that $\binom{m-1}{r-1} \Big/ \binom{n-1}{r-1}$ is $\downarrow$ in r. Now note that

$$\frac{q_{n+2} \cdots q_{n+m}}{q_1 \cdots q_{n-1}} \geq \frac{p_{n-1} q_{n+3} \cdots q_{n+m}}{p_{n+2} q_1 \cdots q_{n-2}}$$

since $p_{n-1} \geq p_{n+2} \geq \frac{1}{2}$, which implies that $p_{n+2} q_{n+2} \geq p_{n-1} q_{n-1}$. It follows that $C_1 \geq C_2$, and similarly one can show that $C_2 \geq C_3 \geq \dots \geq C_m$. ||

Theorem 2.6. Let $p_{21} \geq p_{22}$ and $p_1 \geq \dots \geq p_{n+m} \geq \frac{1}{2}$. A sufficient condition for $N \overset{st}{\geq} N_\sigma^\Pi$ for all permutations σ and Π of $\{1, 2\}$ and $\{1, 2, \dots, n + m\}$ respectively is that

$$\left(\frac{p_{21}}{1-p_{21}}\right)\left(\frac{p_{22}}{1-p_{22}}\right) \geq \frac{q_{n+2}\cdots q_{n+m}}{q_1\cdots q_{n-1}} \quad . \qquad (2)$$

<u>Proof</u>. a) We show initially that if (2) is satisfied, then $N \overset{st}{\geq} N^{\Pi}$ for all Π.

Let A^{Π} be a given arrangement or allocation. We can without loss of generality assume that $\Pi(1) < \ldots < \Pi(n)$ and $\Pi(n+1) < \ldots < \Pi(n+m)$. If Π is not the identity, then $\Pi(n+1) < \Pi(n)$ and we define Π' by $\Pi'(i) = \Pi(i)$ for $i \notin \{n, n+1\}$, $\Pi'(n) = \Pi(n+1)$, $\Pi'(n+1) = \Pi(n)$. Since (2) is satisfied and $C_1 = \dfrac{q_{n+2}\cdots q_{n+m}}{q_1\cdots q_{n-1}}$, it follows from Lemmas 2.1, 2.3, 2.4, and 2.5 that $N^{\Pi'} \overset{st}{\geq} N^{\Pi}$. We proceed now in this fashion where at each new step we obtain a new arrangement which is stochastically superior to the previous one until we obtain a permutation Π^* such that $\Pi^*(i) \leq n$ for all $i = 1, \ldots, n$. In other words $N = N^{\Pi^*} \overset{st}{\geq} \ldots \overset{st}{\geq} N^{\Pi'} \geq N^{\Pi}$.

b) We now show that $N \overset{st}{\geq} N_{\sigma}^{\Pi}$ for any Π and σ where $\sigma(1) = 2$ and $\sigma(2) = 1$. We want to show that

$$N = X_{21}(X_1+\ldots+X_n)+X_{22}(X_{n+1}+\ldots+X_{n+m}) \overset{st}{\geq} X_{22}(X_{\Pi(1)}+\ldots+X_{\Pi(n)})$$
$$+ X_{21}(X_{\Pi(n+1)}+\ldots+X_{\Pi(n+m)}) = N_{\sigma}^{\Pi}.$$

It suffices to show that for $1 \leq r \leq n$,

$$p_{21}(1-p_{22})[X_1+\ldots+X_n \geq r]+p_{22}(1-p_{21})[X_{n+1}+\ldots+X_{n+m} \geq r]$$
$$\geq p_{22}(1-p_{21})[X_{\Pi(1)}+\ldots+X_{\Pi(n)} \geq r]+p_{21}(1-p_{22})[X_{\Pi(n+1)}+\ldots+X_{\Pi(n+m)} \geq r],$$

or equivalently that

$$\left(\frac{p_{21}}{1-p_{21}}\right)\Big/\left(\frac{p_{22}}{1-p_{22}}\right) \geq \frac{[X_{\Pi(1)}+\ldots+X_{\Pi(n)}\geq r]-[X_{n+1}+\ldots+X_{n+m}\geq r]}{[X_1+\ldots+X_n\geq r]-[X_{\Pi(n+1)}+\ldots+X_{\Pi(n+m)}\geq r]}$$

But the right hand side of the last expression is ≤ 1 and $p_{21} \geq p_{22}$ from which the result follows. ||

<u>Corollary 2.7</u>. Let $p_{21} \geq p_{22}$ and $p_1 \geq \cdots \geq p_n > p_{n+1} \geq \cdots \geq p_{n+m} \geq \frac{1}{2}$. A necessary and sufficient condition for $N \overset{st}{\geq} N_\sigma^\Pi$ for all permutations σ and Π of $\{1, 2\}$ and $\{1, \ldots, n+m\}$ is that

$$\left(\frac{p_{21}}{1-p_{21}}\right)\Big/\left(\frac{p_{22}}{1-p_{22}}\right) \geq \frac{q_{n+2}\cdots q_{n+m}}{q_1\cdots q_{n-1}} . \tag{3}$$

<u>Proof</u>. By Theorem 2.6 the condition is sufficient. Consider now the arrangement A_σ^Π where $\sigma(1) = 1$, $\Pi(i) = i$ if $i \notin \{n, n+1\}$, and $\Pi(n) = n+1$. If $N \geq N_\sigma^\Pi$ for this Π and σ, then

$$[N = 0] \leq [N_\sigma^\Pi = 0]$$

or

$$p_{21}(1-p_{22})[q_1\cdots q_n - q_1\cdots q_{n-1}q_{n+1}] \leq p_{22}(1-p_{21})[q_nq_{n+2}\cdots q_{n+m} - q_{n+1}\cdots q_{n+m}]$$

or

$$\left(\frac{p_{21}}{1-p_{21}}\right)\Big/\left(\frac{p_{22}}{1-p_{22}}\right) \geq \frac{q_{n+2}\cdots q_{n+m}}{q_1\cdots q_{n-1}} \quad \text{since } q_n - q_{n+1} < 0. \ ||$$

Remark 2.8. Theorem 2.6 and Corollary 2.7 clearly show that when generator G_1 is sufficiently better than generator G_2 (to the extent that

$$\left(\frac{p_{21}}{1-p_{21}}\right)\Big/\left(\frac{p_{22}}{1-p_{22}}\right) \geq \frac{q_{n+2}\cdots q_{n+m}}{q_1\cdots q_{n-1}}\Bigg), \text{ then}$$

one can do no better than to allocate G_1 and the n "best" machines to location 1.

Example 2.9. Location 1 needs 3 machines and location 2 needs 2. Suppose that $p_{21} = .99$ and $p_{22} = .88$ are the respective probabilities of the two generators functioning at some future time t_0, while $p_1 = .88$, $p_2 = .86$, $p_3 = .84$, $p_4 = .82$, and $p_5 = .80$ are the respective probabilities for the machines. Since

$\left(\frac{.99}{.01}\right)\Big/\left(\frac{.88}{.12}\right) = 13.5 \geq 11.9 = \frac{.20}{(.12)(.14)}$, we can do no better than to allocate G_1, M_1, M_2, and M_3 to location 1 if we are interested in maximizing the number of operable machines at time t_0.

Example 2.10. Suppose $n = m = 5$, $p_{21} = .90$, and $p_{22} = .75$. If $p_i \in [.9, .92]$ for all $i = 1, \ldots, 10$, then $N \overset{st}{\geq} N^{\Pi}_{\sigma}$ for all Π and σ since

$$\left(\frac{p_{21}}{1-p_{21}}\right)\Big/\left(\frac{p_{22}}{1-p_{22}}\right) = \left(\frac{.9}{.1}\right)\left(\frac{.75}{.25}\right) = 3 \geq \left(\frac{.10}{.08}\right)^4 \geq \frac{q_7q_8q_9q_{10}}{q_1q_2q_3q_4}\ .$$

Example 2.11. Suppose $n = 4$, $m = 3$, and $p_i \in [.9, .92]$ for all $i = 1, \ldots, 7$ (that is all the machines have reliability at time t_0 in the interval $[.9, .92]$). In this case,

$$\frac{q_6 q_7}{q_1 q_2 q_3} \le \frac{(.1)^2}{(.08)^3} = 19.5.$$

Hence we see that in order for N to correspond to the stochastically best arrangement, $\left(\frac{p_{21}}{1-p_{21}}\right) \Big/ \left(\frac{p_{22}}{1-p_{22}}\right)$ must be rather large. If $p_{21} = .9$ and $p_{22} = .75$ then this is not the case, although if $p_{21} = .99$ and $p_{22} = .75$ this is true.

References

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